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A Way to Simplify Truth Functions

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A WAY TO SIMPLIFY TRUTH FUNCTIONS

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The quest is herewith resumed of a convenient technique for converting a truth-functional formula into its shortest equivalent in alternational normal form. We may, as before,* confine our attention to formulas given in alternational normal form. These may be described as comprising all *literals* (letters and negations of letters), and more generally all *fundamental formulas* (literals and conjunctions of literals, containing no letter twice), and more generally all alternations of fundamental formulas.

It will also be convenient, for purposes of the technique here to be developed, to exclude from consideration those formulas which are valid, or tautologous. A formula can be quickly tested for validity, and if found valid it can be rewritten in simplest form, '$p \vee \bar{p}$', out of hand. So let us assume hereafter that the formulas under investigation are in alternational normal form but not valid.

Where ϕ and ψ are fundamental formulas, we say that ϕ *subsumes* ψ if all the literals whereof ψ is a conjunction are among the literals whereof ϕ is a conjunction. We call ϕ a *prime implicant* of a formula Φ if ϕ implies Φ and subsumes no shorter formula which implies Φ. Now any shortest equivalent (in alternational normal form, as usual) of a formula Φ is an alternation of prime implicants of Φ (PSTF, p. 524); so a major part of the job of finding a shortest equivalent of Φ is the eliciting of all the prime implicants of Φ. A drawback of the procedure in PSTF, there remarked upon (p. 531), was that the prime implicants were exhausted only with help of a preliminary expansion into the cumbersome "developed normal form." Now, on the other hand, a speedy and direct method will be explained for getting all the prime implicants. Φ can be transformed into the alternation of all its prime implicants simply by continued use of the following operations (i) and (ii).**

(i) *Drop these obvious superfluities:* If one of the clauses of alternation subsumes another, drop the subsuming clause. Also supplant $\alpha \vee \bar{\alpha}\phi$ by $\alpha \vee \phi$ (and $\bar{\alpha} \vee \alpha\phi$ by $\bar{\alpha} \vee \phi$), where α is a single letter.

(ii) *Adjoin, as an additional clause of alternation, the* consensus *of two clauses.* Definition: The conjunction $\phi\psi$ (with any duplicate literals deleted) is called the *consensus* of $\alpha\phi$ and $\bar{\alpha}\psi$, provided that it contains no letter both affirmed and negated. The operation (ii) is to be regarded as not applying in case the

* *The problem of simplifying truth functions*, this MONTHLY, vol. 59, 1952, pp. 521–531; cited hereafter as PSTF, but not drastically presupposed.

** *Note added June 7, 1955:* It has today come to my attention, more than two months after submission of the present paper, that this result was anticipated in an Air Force memorandum of April 1954 by Edward W. Samson and Burton E. Mills, *Circuit minimization: algebra and algorithms for new Boolean canonical expressions*, AFCRC Technical Report 54-21. For my present paper I would still plead brevity and perspicuity, and certain novelties in the later portion; but to Samson and Mills belongs the credit for discovering that the alternation of prime implicants can be got by (i) and (ii).

consensus subsumes a clause already present; otherwise we could get an unending oscillation of (i) and (ii).

The two operations are to be performed as long as possible. (i), in particular, is to be performed as much as possible before and after each performance of (ii). When neither is applicable further, then, as will be proved, we have the alternation of all and only the prime implicants. First let us see an example.

(1) $$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee prs \vee qr\bar{s} \vee pqrt.$$

By (i) we drop '*prs*', which subsumes '*ps*'. By (ii), next, we add on the consensus '*pqr*' of '*ps*' and '$qr\bar{s}$'. Result:

(2) $$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s} \vee pqrt \vee pqr.$$

By (i) now we drop '*pqrt*', which subsumes '*pqr*'. Result:

(3) $$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s} \vee pqr.$$

By (ii), finally, we add on the consensus '$r\bar{s}t$' of '$\bar{q}t$' and '$qr\bar{s}$', and also the consensus '*prt*' of '$\bar{q}t$' and '*pqr*', obtaining:

(4) $$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s} \vee pqr \vee r\bar{s}t \vee prt.$$

Here the process ends. No further pair of clauses has a consensus, except such as would subsume (indeed match) an existing clause.

Now to the proof that (i) and (ii) deliver all the prime implicants. This will be proved by proving that any (non-valid) formula Φ is still susceptible to an application of (i) or (ii) as long as there is a prime implicant χ of Φ which is not a clause of Φ.

Φ is implied by χ, yet not valid; so χ must have letters in common with Φ. Moreover, since χ is a *prime* implicant, it has no letters foreign to Φ; for, any such could be dropped without impairing the implication.* Moreover, since χ is a prime implicant of Φ, and each clause of Φ also implies Φ, no clause of Φ other than χ itself is subsumed by χ; hence none, since χ is not a clause of Φ. So there is at least one fundamental formula (χ itself, for one) fulfilling these three conditions: (a) it subsumes χ, (b) it subsumes no clause of Φ, and (c) it contains only letters of Φ. Let ψ be a longest fundamental formula fulfilling (a), (b), and (c). Still ψ will lack some letter α of Φ. (For, if ψ contained all letters of Φ, then, by (b), ψ would conflict with each clause of Φ in point of the affirming or negating of some letter or other; whereas we know rather, by (a), that ψ implies Φ.) Now since ψ is a longest formula fulfilling (a), (b), and (c), the longer formulas $\alpha\psi$ and $\bar{\alpha}\psi$ must fail to fulfill (b); for they do fulfill (a) and (c). So $\alpha\psi$ and $\bar{\alpha}\psi$ each subsumes a clause of Φ. These subsumed clauses must contain α and $\bar{\alpha}$ respectively, since they were not subsumed by ψ alone. But these clauses

* The reasoning in this sentence and the preceding one is an improvement on the proof of Theorem 2 in PSTF, where the case of prime implicants consisting of a single literal was overlooked. Strictly speaking, Theorem 2 fails for tautologies, since any literal is by definition a prime implicant of any tautology.

are not simply α and $\bar{\alpha}$, or Φ would be valid. So there are just three possible cases: the clauses are respectively $\alpha\phi$ and $\bar{\alpha}$, where ψ subsumes ϕ (Case 1); or they are α and $\bar{\alpha}\phi'$, where ψ subsumes ϕ' (Case 2); or they are $\alpha\phi$ and $\bar{\alpha}\phi'$ (Case 3). In Case 1, however, Φ contains $\bar{\alpha} \vee \alpha\phi$ and is accordingly susceptible to operation (i). Similarly for Case 2. In Case 3, finally, Φ is susceptible to an application of (ii), consisting in the adding on of the consensus of $\alpha\phi$ and $\bar{\alpha}\phi'$. This consensus, namely $\phi\phi'$ (minus any duplicate literals), is readily seen to meet the requirements of (ii): it contains no letter both affirmed and negated, since it is subsumed by a fundamental formula ψ; and it subsumes no clause of Φ, since ψ subsumed none.

This completes the proof that (i) and (ii) deliver *all* the prime implicants. The proof of the converse, namely that (i) and (ii) when continued as long as possible yield an alternation of prime implicants *only*, can now be added in a few words. Since $\alpha\phi \vee \bar{\alpha}\psi$ is equivalent to $\alpha\phi \vee \bar{\alpha}\psi \vee \phi\psi$, clearly (ii) is, like (i), an equivalence transformation. Accordingly the alternation obtained from a formula Φ by applying (i) and (ii) as long as possible can be depended upon to be an alternation of clauses each of which implies Φ. But, as we just finished proving in the preceding paragraph, every prime implicant is a clause. Accordingly any clause that is not a prime implicant subsumes another clause which is a prime implicant. But this cannot happen; (i) would be applicable again.

I shall discuss, for the remainder of the paper, the business of moving from the alternation of all prime implicants to a shortest equivalent (as always, in alternational normal form). This is wholly a matter of dropping dispensable clauses. There is, moreover, this *test of dispensability* of a single clause ϕ: see whether ϕ implies the remainder, Ψ, of the alternation. This may be quickly decided by testing Ψ for truth when the letters affirmed in ϕ are marked true and those negated in ϕ are marked false.

The dropping of one dispensable clause, however, can render another originally dispensable clause indispensable to the remaining alternation. We want rather to find the largest simultaneously dispensable combination of clauses. Certain aids to this end will now be noted.

We can get a head start by reviewing the applications of (ii) which were made in arriving at the alternation of all prime implicants. We bracket out, in a body, all clauses that were added by (ii) after the last application of (i). In our example we get:

$$(5) \qquad ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s} \vee pqr \vee [r\bar{s}t \vee prt].$$

These brackets serve as a reminder that the clauses enclosed are dispensable, and not only singly but jointly; for, the alternation which includes those bracketed clauses was originally got, from itself minus those clauses, by an equivalence transformation.

We must not, however, without further check, bracket a clause ϕ ('pqr', in the example) which was added by (ii) prior to a use of (i); for the subsequent use of (i) may, by banishing another clause, have rendered ϕ indispensable.

The next move, rather, is to subject each unbracketed clause to the test of dispensability formulated above, and bracket individually each clause which meets the test. The result, in the case of (5), is:

$$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee [qr\bar{s}] \vee [pqr] \vee [r\bar{s}t \vee prt]. \tag{6}$$

This done, we can conclude a good deal about any shortest equivalent. It is bound to retain the unbracketed part. (This is what was called the *core* in PSTF, p. 527.) Its remaining clauses, if any, will be certain of those shown in brackets; and not all of them, since any one bracketed portion, at least, is dispensable. If, as commonly happens, there are no bracketings or just one, then the shortest equivalent is the core itself. It is only in cases such as (6), with an unusual lot of bracketing, that any serious exhaustion of possibilities remains to be done. An approach to such cases (doubtless susceptible, however, to streamlining) is the following.

A bracketed clause ϕ may or may not imply the core; all we know is that ϕ implies the whole alternation minus ϕ itself. If ϕ does imply the core, then ϕ should be cancelled for good. Any shortest equivalent, retaining the whole core as it does, is bound to omit any clause that implies the core. Such clauses are *absolutely* dispensable—independently of the omission or retention of other bracketed clauses.

So each bracketed clause should be tested individually (in the quick way lately mentioned) to see if it implies the core; and each which does should be deleted. In our example (6), none of the four bracketed clauses proves to imply the core. But here is an example where the phenomenon does occur:

$$pq \vee pr \vee \bar{p}s \vee \bar{r}t \vee [pt] \vee [qs].$$

Here, in fact, each of the bracketed prime implicants proves absolutely dispensable; only the core remains.

In general the remaining task, if any, in finding the shortest equivalents of a formula, is to test combinations of bracketed passages for joint dispensability. This can be done as follows. To see whether $\phi_1, \cdots, \phi_n$ are jointly dispensable in $\Psi \vee \phi_1 \vee \cdots \vee \phi_n$, hence whether $\phi_1 \vee \cdots \vee \phi_n$ implies Ψ, check separately for each i (by the quick method noted earlier) to see whether ϕ_i implies Ψ. If ϕ_i implies Ψ for each i, then and only then $\phi_1 \vee \cdots \vee \phi_n$ implies Ψ and is dispensable *in toto*.

There is evident strategy in testing big combinations ahead of smaller ones. Only partial combinations want testing, however; the preceding search for absolutely dispensable clauses was *ipso facto* a test of whether the *whole* bracketed portion was dispensable.

In the case of (6), what we find is that '$qr\bar{s} \vee r\bar{s}t \vee prt$' implies the alternation of the remaining four clauses, and similarly for '$pqr \vee r\bar{s}t \vee prt$'; so we end up with two shortest equivalents:

$$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee pqr, \qquad ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s}.$$

Despite the evident advantage of our new method over the method of developed normal forms and tables in PSTF, we should continue to exploit the separation expedient noted on page 529 of PSTF when we can.

ON SPHERICAL DRAWING AND COMPUTATION

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1. Introduction. Pictorial spherical diagrams (Figure 1) are tedious to draw by conventional methods and, as a result, are frequently found approximated or sketched, even in textbooks and treatises. Such diagrams will be recognized as axonometric projections of a sphere intersected by planes passing through its center. Besides their pictorial value, such diagrams provide a means for the geometrical solution of spherical triangles.

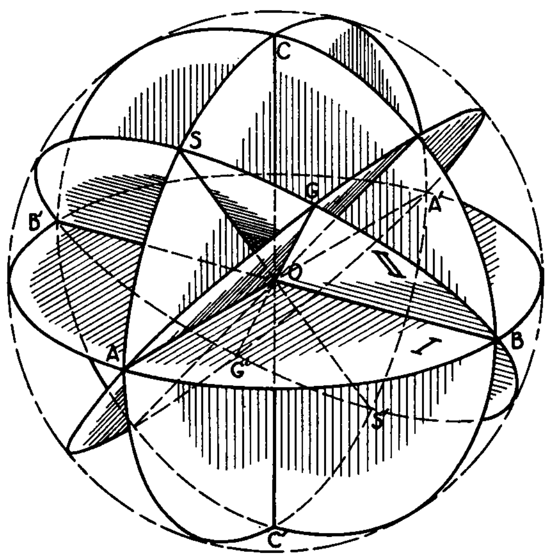

FIG. 1

The planes shown in Figure 1 intersect the sphere in great circles, and these appear on the drawing as *derived ellipses* of the respective great circles *primitive* to them. Orthogonal diameters of the great circles appear as *conjugate diameters* of the derived ellipses, for example, AA', CC'. The dihedral angle between two planes (I and II) having a given line of intersection (BB') is measured by the intercepted arc (AS) of the great circle ($ACA'C'$) normal to BB'.

Two characteristic problems are involved in drawing diagrams like Figure 1: