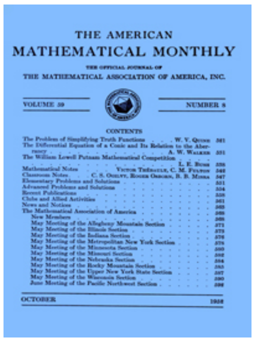

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THE PROBLEM OF SIMPLIFYING TRUTH FUNCTIONS

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The formulas of the propositional calculus, or the logic of truth functions, are here to be understood as built up of the statement letters 'p', 'q', 'r', $\cdots$ by just the notations of negation, conjunction, and alternation (or disjunction), *viz.* '$\bar{p}$', 'pq', and '$p \vee q$', to any degree of iteration. A formula is *valid* if it comes out true under all assignments of truth values to the letters, and *consistent* if it comes out true under some. One formula *implies* another if there is no assignment of truth values which makes the first formula true and the second false. Two formulas are *equivalent* if they imply each other. Implication and equivalence, so defined, are relations of formulas; they are not to be confused with the conditional and biconditional, commonly expressed by '$\supset$' and '$\equiv$'. These latter notations will be omitted, being translatable into terms of negation, conjunction, and alternation in familiar fashion.

It will be convenient to use the words 'conjunction' and 'alternation' in slightly extended senses. Ordinarily one speaks of a conjunction of two or more formulas; but I shall speak also of a conjunction of one formula, meaning thereby simply the formula itself. Thus every formula is a conjunction at least of itself. Correspondingly for alternation.

Letters and negations of letters will be spoken of collectively as *literals*. A conjunction of literals will be called a *fundamental* formula if no letter appears in it twice. Literals themselves count as fundamental formulas, in view of my broad use of the word 'conjunction'. Finally any alternation of fundamental formulas will be called a *normal* formula, and the fundamental formulas of which it is an alternation will be called its *clauses*. Fundamental formulas themselves count as normal, in view of my broad use of the word 'alternation'; a fundamental formula is a one-clause normal formula. In general, thus, normal formulas are simply what have been known in the literature as *disjunctive normal forms*, or *alternational normal forms*, except that normal formulas are subject to one additional requirement: no letter can occur in a clause twice. A normal formula cannot contain 'pqp' nor '$\bar{p}q\bar{p}$' nor '$pq\bar{p}$'. Mechanical routines are well known for transforming any consistent formula into an equivalent which is normal.*

But there remains a problem which, despite the trivial character of truth-function logic, has proved curiously stubborn; *viz.*, the problem of devising a general mechanical procedure for reducing any formula to its *simplest* equivalent. Since Shannon's correlation of the formulas of truth-function logic with electric circuits,† this problem of simplification has taken on significance for engineering; for, a technique for simplifying truth-functional formulas would be

* See, *e.g.*, §10 of my *Methods of Logic*, Henry Holt & Co., 1950.

† C. E. Shannon, A symbolic analysis of relay and switching circuits, Trans. Amer. Inst. of Electrical Engineers, vol. 57, 1938, pp. 713–723.

a technique for simplifying circuits. It is noteworthy that the staff of the Computation Laboratory of Harvard University have found it worth while to set forth elaborate procedures for simplifying truth functions, and even to tabulate all the simplest equivalents of formulas involving four or fewer letters.‡

In a certain theoretical sense, indeed, there is no problem. Given any formula, we can, in principle, survey the totality of simpler formulas involving no additional letters; for this totality is finite. By truth tables or otherwise, we can test each of these simpler formulas for equivalence to the given formula, and thus pick out the simplest equivalent. This procedure is mechanical; what is wanted, however, is a mechanical procedure which is short enough to be practical.

Because of the perspicuity and general convenience of normal formulas, an interesting specialization of the simplification problem is the problem of finding a simplest *normal* equivalent. In fact we may limit our problem to normal formulas from start to finish, since the preliminary step of converting a given formula into *some* normal equivalent, not necessarily a simplest, presents no problem. By limiting our consideration thus to normal formulas we are indeed disregarding inconsistent formulas, but this is no real limitation, since a shortest equivalent of any inconsistent formula can be supplied out of hand: '$p\bar{p}$'. So the problem which I shall examine is that of converting any normal formula into a simplest normal equivalent. This is not the most general form of the simplification problem from the point of view of engineering, since it can happen that some short non-normal formula represents a still cheaper electric circuit than any normal equivalent. But it will be more than enough to occupy us on the present occasion.

Limiting ourselves to normal formulas, we still have some choice as to our measure of simplicity. We might simply count all occurrences of literals and alternation signs, or we might put a premium on fewness of clauses and so resort to a count of occurrences of literals only when comparing formulas which are alike in number of clauses. What I shall have to say in this paper will not require any decision, however, between these or other reasonable standards of simplicity.

Let us use the Greek letter 'ζ' to refer to any literal, and 'ϕ', 'ψ', 'χ' to refer to any fundamental formulas, and 'Φ' and 'Ψ' more generally to refer to any normal formulas. In order to refer to compounds of formulas which are severally referred to thus by Greek letters, let us use corresponding compounds of the Greek letters themselves; thus where ζ is taken as '$\bar{p}$' and ϕ as '$p\bar{q}$', $\zeta \vee \phi$ is to be understood as '$\bar{p} \vee p\bar{q}$'.

Now it can happen that some clause ϕ is superfluous in a normal formula $\phi \vee \Psi$; *i.e.*, that $\phi \vee \Psi$ is equivalent to Ψ alone. It can also happen that an occurrence of a literal ζ is superfluous in a normal formula $\phi\zeta \vee \Psi$; *i.e.*, that $\phi\zeta \vee \Psi$ is equivalent to $\phi \vee \Psi$ alone. Weeding out such superfluous clauses and literals is

‡ *Synthesis of Electronic Computing and Control Circuits*, by the aforementioned staff, headed by Howard H. Aiken. Harvard University Press, 1951.

the obvious way of reducing normal formulas to simpler normal equivalents. To implement this sort of reduction, all we need are convenient techniques for spotting superfluous clauses and literals. Now such techniques are readily devised, as follows.

To say that $\phi \vee \Psi$ is equivalent to Ψ is the same as saying that ϕ implies Ψ. Also, as is slightly less evident but readily verifiable, to say that $\phi\zeta \vee \Psi$ is equivalent to $\phi \vee \Psi$ is the same as saying that ϕ implies $\zeta \vee \Psi$. To test a clause ϕ for superfluousness in a normal formula, therefore, we have only to see whether ϕ implies the rest of the normal formula; and to test an occurrence of a literal ζ in a clause $\phi\zeta$ of a normal formula for superfluousness, we have again only to see whether ϕ implies the rest of the normal formula. Now the ϕ in either problem is a fundamental formula; and any question of implication on the part of a fundamental formula ϕ is always quickly settled. To find whether ϕ implies any given formula we have merely to mark as true, throughout the given formula, all the letters which occur affirmatively in ϕ, and as false all the letters which occur negated in ϕ, and then see whether the given formula thereupon comes out true (for all values of any remaining letters).

Example 1: We find the clause '$p\bar{r}$' of '$pq \vee p\bar{r} \vee \bar{q}\bar{r}$' superfluous by testing to see if it implies the rest, '$pq \vee \bar{q}\bar{r}$'. The test of implication consists in putting 'T' for 'p' and 'F' for 'r' (conformably with '$p\bar{r}$') in '$pq \vee \bar{q}\bar{r}$'; the result is '$\mathrm{T}q \vee \bar{q}\mathrm{T}$', which reduces to '$q \vee \bar{q}$'.

Example 2: We find the first occurrence of '$\bar{q}$' in '$pq \vee p\bar{q}r \vee \bar{p}\bar{q}\bar{r}$' superfluous by testing to see if 'pr' implies the rest, '$pq \vee \bar{q} \vee \bar{p}\bar{q}\bar{r}$'. The test of implication consists in putting 'T' for 'p' and 'r' in '$pq \vee \bar{q} \vee \bar{p}\bar{q}\bar{r}$'; the result '$\mathrm{T}q \vee \bar{q} \vee \mathrm{F}\bar{q}\mathrm{F}$' reduces to '$q \vee \bar{q}$'.

Let us call a normal formula *irredundant* if it has no superfluous clauses and none of its clauses has superfluous literals. We now have a mechanical routine for reducing any normal formula to an irredundant equivalent. Summed up, it runs as follows. First try each clause in turn to see whether it implies the rest of the formula; whenever any clause is found which does imply the rest of the formula, delete it once and for all before continuing the survey. After all reductions of this type are at an end, then try each "immediate subclause" (a clause minus one of its literals) to see whether it implies the rest of the formula; if it does, delete the superfluous occurrence of the literal. When this process can be carried no farther, we have an irredundant formula.

It seems reasonable to hope that this procedure of simplification, issuing as it does in a normal equivalent which is irredundant, may solve our original problem; namely, the problem of reducing any normal formula to a simplest normal equivalent. The procedure leads to a normal formula in which no clause is superfluous and no occurrences of literals within clauses are superfluous; and it seems reasonable to suppose that such a normal formula is as simple as any equivalent normal formula can be.

But this is not so. Consider the normal formula '$p\bar{q} \vee \bar{p}q \vee q\bar{r} \vee \bar{q}r$'. This is irredundant; no clause can be dropped, nor can any occurrences of literals be dropped, without breach of equivalence. Yet this formula has simpler normal equivalents, indeed two: both '$p\bar{q} \vee \bar{p}r \vee q\bar{r}$' and '$\bar{p}q \vee p\bar{r} \vee \bar{q}r$', as can be checked by truth tables. These are simpler than the original by any conceivable standard of simplicity, but they cannot be got from the original by any process of dropping or curtailing clauses.

The routine of eliminating redundancies by dropping or curtailing clauses remains useful, for it is quick and easy and it brings gains in simplicity wherever it can be used. But it does not assure us always of a simplest result. The remainder of this paper will be devoted to presenting a general procedure for finding a really simplest normal equivalent. The procedure will be laborious, but not to the point of unmanageability.

A normal formula is called *developed* if all of its letters appear in each of its clauses; e.g., '$pq\bar{r} \vee p\bar{q}r$'. Any normal formula can be turned into a developed equivalent by an obvious procedure: any clause ϕ which lacks a letter ζ can be supplanted by its equivalent $\phi\zeta \vee \phi\bar{\zeta}$, and the process can be continued until each clause contains each letter, duplicate clauses being dropped as they arise. Example: the normal formula '$pqr \vee r\bar{s}$' becomes '$pqrs \vee pqr\bar{s} \vee pr\bar{s} \vee \bar{p}r\bar{s}$', which in turn becomes '$pqrs \vee pqr\bar{s} \vee p\bar{q}r\bar{s} \vee \bar{p}qr\bar{s} \vee \bar{p}\bar{q}r\bar{s}$', a developed normal formula. The procedure for finding simplest normal equivalents will take developed normal formulas as its point of departure. Meanwhile a couple of auxiliary notions must be defined, and their properties established.

DEFINITIONS: ϕ will be said to *subsume* ψ if and only if all the literals whereof ψ is a conjunction are among the literals whereof ϕ is a conjunction. ϕ will be called a *prime implicant* of Ψ if and only if ϕ implies Ψ and subsumes no shorter formula which implies Ψ. ϕ will be called a *completion* of χ with respect to Ψ if and only if ϕ subsumes χ and contains all letters of Ψ and no others.

THEOREM 1. *Any simplest normal equivalent of* Φ *is an alternation of prime implicants of* Φ.

Proof. Every clause ψ of a normal equivalent Ψ of Φ implies Ψ and therefore Φ. So, if ψ is not a prime implicant of Φ, then ψ subsumes a shorter formula ψ' which implies Φ and therefore Ψ. But then Ψ has one or more redundant occurrences of literals, in ψ, which could be deleted (as noted in earlier pages); so Ψ is not a simplest normal equivalent.

The above theorem brings out the relevance, to our simplification project, of listing the prime implicants of a formula Φ. The way to obtain such a list will become evident, in the case of developed Φ, after the next three theorems.

THEOREM 2. *No prime implicant of* Φ *contains letters foreign to* Φ.

Proof. If $\psi\zeta$ implies Φ and the letter in ζ is foreign to Φ, then any assignment of truth values which makes ψ true will make Φ true, regardless of ζ; *i.e.*, ψ will imply Φ, and hence $\psi\zeta$ will not be a prime implicant of Φ.

THEOREM 3. *If Φ is a developed normal formula and contains all letters of ψ, then ψ implies Φ if and only if all completions of ψ with respect to Φ are clauses of Φ.*

Proof. If ψ has a completion ψ' which is not a clause of Φ, then each clause of Φ contains a letter affirmatively which is negated in ψ' or vice versa. Then the assignment of truth values to letters which makes ψ' true makes all clauses of Φ false, though making ψ true; so ψ does not imply Φ. Conversely, each assignment of truth values to the letters of Φ which makes ψ true makes some completion of ψ true; so, if all the completions of ψ are clauses of Φ, then each assignment which makes ψ true makes Φ true, and hence ψ implies Φ.

From Theorems 2 and 3 and the definition of prime implicant, there follows this corollary:

THEOREM 4. *ψ is a prime implicant of a developed normal formula Φ if and only if all letters of ψ are among those of Φ and all completions of ψ with respect to Φ are clauses of Φ and there is no shorter formula ψ', subsumed by ψ, such that all completions of ψ' with respect to Φ are clauses of Φ.*

Theorem 4 enables us, given a developed normal formula $\phi_1 \vee \cdots \vee \phi_n$, to arrive at its prime implicants by the following mechanical routine. We make a growing list which does not begin as a list of prime implicants, but begins rather with $\phi_1, \cdots, \phi_n$ and is extended according to the following principle: whenever two entries can be found in the list which are related as $\chi\zeta$ and $\chi\bar{\zeta}$ (thus identical except for a negation sign), add their common part χ as a new entry in the list. Check marks are to be applied to any entries $\chi\zeta$ and $\chi\bar{\zeta}$ which thus generate new entries, bût a check mark is not to be treated as disqualifying an entry from reuse; thus '$pqrs$' can be used once with '$pqr\bar{s}$' to generate 'pqr' and once with '$p\bar{q}rs$' to generate 'prs'. When the list has been extended as far as possible by the above process, we can read off the prime implicants of $\phi_1 \vee \cdots \vee \phi_n$ from it thus: they are the entries which bear no check marks.

Example: Suppose $\phi_1, \cdots, \phi_n$ are '$pqrs$', $\bar{p}\bar{q}rs$', '$pq\bar{r}s$', '$p\bar{q}\bar{r}s$', '$pq\bar{r}\bar{s}$', and '$p\bar{q}\bar{r}\bar{s}$'. The first and third of these six yield 'pqs' as a seventh entry in our list; the third and fourth yield '$p\bar{r}s$' as an eighth; the third and fifth yield '$pq\bar{r}$' as a ninth; the fourth and sixth yield '$p\bar{q}\bar{r}$' as a tenth; and the fifth and sixth yield '$p\bar{r}\bar{s}$' as an eleventh. Of the original six entries, all but the second receive check marks in the process. Proceeding now to generate still further entries from the five added ones, we get '$p\bar{r}$' twice and nothing more; and accordingly we apply check marks to '$p\bar{r}s$' and '$p\bar{r}\bar{s}$', and also to '$pq\bar{r}$' and '$p\bar{q}\bar{r}$'. (Note the necessity of applying check marks to all four, despite the duplicate nature of their yield.) Surveying the finished list, we find just these entries without checks: '$\bar{p}\bar{q}rs$', 'pqs', '$p\bar{r}$'. These are the prime implicants of '$pqrs \vee \bar{p}\bar{q}rs \vee pq\bar{r}s \vee p\bar{q}\bar{r}s \vee pq\bar{r}\bar{s} \vee p\bar{q}\bar{r}\bar{s}$'.

How to use the list of prime implicants, in order to obtain a simplest normal equivalent of a developed normal formula, is suggested by the next theorem together with Theorem 1.

THEOREM 5. *If Ψ is a simplest normal equivalent of a developed normal formula Φ, then each clause of Φ subsumes a clause of Ψ.*

Proof. Consider any clause ϕ of Φ. By Theorems 1 and 2, Ψ contains no letters foreign to Φ, nor, therefore, to ϕ. Hence any clause of Ψ which ϕ does not subsume must contain a letter affirmatively which is negated in ϕ or vice versa. Hence the assignment of truth values to letters which makes ϕ true will make all clauses of Ψ false except those which ϕ subsumes. Hence ϕ must subsume a clause of Ψ if ϕ is to imply Ψ. But ϕ does imply Ψ, since Φ is equivalent to Ψ.

In view of Theorems 1 and 5, we can obtain a panorama of all simplest normal equivalents of a developed normal formula $\phi_1 \vee \cdots \vee \phi_n$ as follows. First we list the prime implicants, as seen earlier. Then we survey the various subsets of the list, such that ϕ_i for each i subsumes a member of the subset. Each simplest such subset, written as an alternation, is a simplest normal equivalent of $\phi_1 \vee \cdots \vee \phi_n$.

The survey is facilitated by constructing what I shall call the *table of prime implicants* of $\phi_1 \vee \cdots \vee \phi_n$. The abscissas of the table, inscribed across the top, are $\phi_1, \cdots, \phi_n$. The ordinates of the table, inscribed down the left side, are the prime implicants of $\phi_1 \vee \cdots \vee \phi_n$. In the interior of the table we enter crosses in those positions whose abscissas subsume their ordinates.

For the example '$pqrs \vee \bar{p}\bar{q}rs \vee pq\bar{r}s \vee p\bar{q}\bar{r}s \vee pq\bar{r}\bar{s} \vee p\bar{q}\bar{r}\bar{s}$', whose prime implicants were derived earlier, the table is this:

	$pqrs$	$\bar{p}\bar{q}rs$	$pq\bar{r}s$	$p\bar{q}\bar{r}s$	$pq\bar{r}\bar{s}$	$p\bar{q}\bar{r}\bar{s}$
$\bar{p}\bar{q}rs$		×				
pqs	×		×			
$p\bar{r}$			×	×	×	×

Once we have the table of prime implicants, we canvass all ways of so selecting ordinates as to *represent* all abscissas; *i.e.*, to show crosses under all abscissas. We settle upon a selection such that the alternation of the selected ordinates will be as simple as possible. In the above example there is no choice; no selection of rows, short of all three, exhibits crosses in all columns. So in this example the simplest normal equivalent is '$\bar{p}\bar{q}rs \vee pqs \vee p\bar{r}$', which uses all the prime implicants.

For another example let us return to '$p\bar{q} \vee \bar{p}q \vee q\bar{r} \vee \bar{q}r$', which was cited earlier to show that irredundant formulas could have simpler equivalents. To find the simplest normal equivalents of this example by our new general method, we must first expand the formula into a developed normal formula, then derive the list of prime implicants, and finally form the table. The table turns out thus:

	$p\bar{q}r$	$p\bar{q}\bar{r}$	$\bar{p}qr$	$\bar{p}q\bar{r}$	$pq\bar{r}$	$\bar{p}\bar{q}r$
$p\bar{q}$	×	×				
$\bar{q}r$	×					×
$p\bar{r}$		×			×	
$\bar{p}q$			×	×		
$\bar{p}r$			×			×
$q\bar{r}$				×	×	

Survey of the table shows two ways of so picking three rows as to represent all columns, so we come out with two simplest normal equivalents, '$\bar{p}q \vee \bar{q}r \vee p\bar{r}$' and '$p\bar{q} \vee \bar{p}r \vee q\bar{r}$'.

Incidentally the list of prime implicants of a formula Φ has other uses besides its use in obtaining the simplest normal equivalents of Φ. It provides a panorama of all the fundamental formulas which imply Φ; for, the fundamental formulas which imply Φ are simply the prime implicants and all other fundamental formulas which subsume any of them.

So far as concerns the topic of the present paper, however, the use of the table is in finding shortest normal equivalents. As described thus far, the use of the table for this purpose proceeds by exhaustion: trying all the combinations of ordinates which represent all abscissas, and comparing all the resulting alternations for simplicity. Now this process of canvassing the table can be speeded up in many examples (though not in the above two) by the following routine of preparatory reduction.*

(i) If any columns of the table of a formula Φ contain only one cross apiece, then record for future reference the alternation of the ordinates of those crosses. Let us call this alternation the *core* of Φ. (The clauses of the core are bound, by Theorem 5, to be clauses of any simplest normal equivalent of Φ.)

(ii) Reduce the table by deleting the ordinates concerned in (i), and deleting also all abscissas represented by those ordinates. (These abscissas need no further consideration because they will be represented by clauses of our final simplification of Φ anyway as long as we take care to include the core as part of that final simplification.)

(iii) Wherever in the surviving table there are abscissas ϕ_i and ϕ_j such that ϕ_i has crosses only in rows in which ϕ_j has crosses, delete ϕ_j. (For, our final formula is bound to represent ϕ_j anyway, through representing ϕ_i.)

* Note the resemblance of the ensuing operations to the operations on "minimizing charts" which are set forth in pp. 56 ff. of *Synthesis* (see preceding footnote). The clauses of what I call the *core* (below) correspond to what are called "essential combinations" in *Synthesis*. More accurately, the clauses of the core are the duals of the essential combinations; for the minimizing charts produce conjunctional normal forms, in effect, rather than alternation ones. Between the minimizing charts and the tables of the present paper there are profound differences, however, beyond that of duality. A minimizing chart begins as a fixed form which depends only on the multiplicity of letters concerned, and not on the particular formula at hand. It tends in consequence to be more elaborate than the table of prime implicants.

(iv) Delete any ordinates whose crosses have all been lost through the cancelling of abscissas in (ii) and (iii).

The *reduced table of prime implicants* thus achieved can now be subjected to the process, described earlier, of canvassing the ways of selecting ordinates and singling out the most economical. Each end result thus obtained must be supplemented by adjoining the core to it, in alternation.

Example: $pqr \vee p\bar{r} \vee pq\bar{s} \vee \bar{p}r \vee \bar{p}\bar{q}\bar{r}\bar{s}$.
The table of prime implicants turns out as follows:

	$pqrs$	$pqr\bar{s}$	$pq\bar{r}s$	$pq\bar{r}\bar{s}$	$p\bar{q}\bar{r}s$	$p\bar{q}\bar{r}\bar{s}$	$\bar{p}qrs$	$\bar{p}qr\bar{s}$	$\bar{p}\bar{q}rs$	$\bar{p}\bar{q}r\bar{s}$	$\bar{p}\bar{q}\bar{r}\bar{s}$
pq	×	×	×	×							
qr	×	×					×	×			
$p\bar{r}$			×	×	×	×					
$\bar{p}r$							×	×	×	×	
$\bar{p}\bar{q}\bar{s}$										×	×
$\bar{q}\bar{r}\bar{s}$						×					×

Now we apply (i); *i.e.*, observing that the fifth and ninth columns contain only one cross apiece, we record the alternation of the ordinates of those two crosses; *viz.*, '$p\bar{r} \vee \bar{p}r$'. This is the core. Then, applying (ii), we cancel the ordinates '$p\bar{r}$' and '$\bar{p}r$' of the table, and also the eight columns (*viz.*, third through tenth) in which those cancelled rows contained crosses. To what is left of the table, we apply (iii); this enables us to cancel the first or second column at will, say the second. We find no way of applying (iv), so we are now down to our reduced table of prime implicants, which is just this:

	$pqrs$	$\bar{p}\bar{q}\bar{r}\bar{s}$
pq	×	
qr	×	
$\bar{p}\bar{q}\bar{s}$		×
$\bar{q}\bar{r}\bar{s}$		×

Inspection of this table shows just four shortest alternations of ordinates representing both abscissas. They are:

$$pq \vee \bar{p}\bar{q}\bar{s}, \qquad pq \vee \bar{q}\bar{r}\bar{s}, \qquad qr \vee \bar{p}\bar{q}\bar{s}, \qquad qr \vee \bar{q}\bar{r}\bar{s}.$$

Adjoining any of these by alternation to the core '$p\bar{r} \vee \bar{p}r$' gives a simplest normal equivalent of the original formula. We thus end up with four simplest normal equivalents:

$$p\bar{r} \vee \bar{p}r \vee pq \vee \bar{p}\bar{q}\bar{s}, \qquad p\bar{r} \vee \bar{p}r \vee pq \vee \bar{q}\bar{r}\bar{s},$$
$$p\bar{r} \vee \bar{p}r \vee qr \vee \bar{p}\bar{q}\bar{s}, \qquad p\bar{r} \vee \bar{p}r \vee qr \vee \bar{q}\bar{r}\bar{s}.$$

Sometimes the reduced table of prime implicants turns out to be an utter blank, so that the core stands alone as the simplest normal equivalent. An example is '$p\bar{r} \vee \bar{p}\bar{q}r\bar{s} \vee \bar{p}\bar{q}\bar{r}$'. Here the table of prime implicants is:

	$pq\bar{r}s$	$pq\bar{r}\bar{s}$	$p\bar{q}\bar{r}s$	$p\bar{q}\bar{r}\bar{s}$	$\bar{p}\bar{q}r\bar{s}$	$\bar{p}\bar{q}\bar{r}s$	$\bar{p}\bar{q}\bar{r}\bar{s}$
$p\bar{r}$	×	×	×	×			
$\bar{q}\bar{r}$			×	×		×	×
$\bar{p}\bar{q}\bar{s}$					×		×

Applying (i) to this, we obtain '$p\bar{r} \vee \bar{q}\bar{r} \vee \bar{p}\bar{q}\bar{s}$' as core. All rows and columns disappear under (ii), so that we are left with '$p\bar{r} \vee \bar{q}\bar{r} \vee \bar{p}\bar{q}\bar{s}$' itself as the simplest normal equivalent of '$p\bar{r} \vee \bar{p}\bar{q}r\bar{s} \vee \bar{p}\bar{q}\bar{r}$'.

The method admits of one further refinement which, though irrelevant to all the foregoing examples, saves much labor where applicable. I am indebted for it in part to Nelson Goodman. Given a formula for which a simplest normal equivalent is wanted, the new tactic begins by transforming the formula not into a developed normal formula, but rather into an irredundant normal formula by the routine of the early pages of this paper. This irredundant alternation is then separated into as many subsidiary alternations as possible such that no two of them have letters in common. (*E.g.*,'$p\bar{q} \vee \bar{r}s \vee \bar{p}q \vee t$' would be separated into '$p\bar{q} \vee \bar{p}q$', '$\bar{r}s$', and 't'.) Then we expand each of these subsidiary alternations independently into developed normal form and proceed to find a simplest normal equivalent for it, by use of a reduced table of prime implicants as hitherto explained. Finally we make a single alternation of the several results, and this is a simplest normal equivalent of the original formula.

The value of this separation expedient, where applicable, is evident: it saves the exorbitant development of all clauses with respect to all missing letters. But we must prove that the modified method always leads to the simplest normal equivalents. This will be proved as Theorem 8 below; the two intervening theorems are needed as lemmas.

THEOREM 6. *The only irredundant normal formulas which are valid are* '$p \vee \bar{p}$', '$q \vee \bar{q}$', *etc.*

Proof. Let $\phi\zeta \vee \Psi$ be a valid normal formula. Consider then any assignment of truth values to letters which makes ϕ true. It makes $\phi\zeta \vee \Psi$ true, since this is valid; moreover ϕ, being true under this assignment, can be deleted from $\phi\zeta \vee \Psi$ and the result $\zeta \vee \Psi$ will still be true. Thus every assignment which makes ϕ true makes $\zeta \vee \Psi$ true; *i.e.*, ϕ implies $\zeta \vee \Psi$. But this implication was seen, in early pages of the paper, to be the criterion of superfluousness of the occurrence of ζ in $\phi\zeta \vee \Psi$. We see therefore that no valid irredundant normal formula can have the form $\phi\zeta \vee \Psi$. Still every valid normal formula is obviously an alternation of at least two clauses; any single clause is falsifiable. Therefore every valid irredundant normal formula must be an alternation of clauses none of which is of the form $\phi\zeta$; each of which, in other words, is a single literal. Every valid irredundant normal formula has, in short, the form $\zeta_1 \vee \cdots \vee \zeta_n$. But obviously two of $\zeta_1, \cdots, \zeta_n$ must, for validity of $\zeta_1 \vee \cdots \vee \zeta_n$, be negations one of the other. But then each of $\zeta_1, \cdots, \zeta_n$ other than those two is superfluous; or rather there are no others, since $\zeta_1 \vee \cdots \vee \zeta_n$ is supposed to be irredundant. So

$\zeta_1 \vee \cdots \vee \zeta_n$ is just '$p \vee \bar{p}$', or perhaps '$q \vee \bar{q}$', *etc.*

THEOREM 7. *If no two of* $\Phi_1, \cdots, \Phi_n$ *have letters in common, and* ϕ *is a prime implicant of* $\Phi_1 \vee \cdots \vee \Phi_n$, *then* ϕ *contains letters exclusively of* Φ_i *for some* i.

Proof. By Theorem 2, there is an i such that some of the letters of ϕ appear in Φ_i. Now suppose (which will be proved impossible) that there are also letters in ϕ foreign to Φ_i; *i.e.*, that ϕ is $\psi\chi$ where all the letters of ψ but none of the letters of χ are letters of Φ_i. Since ϕ is a prime implicant, neither ψ nor χ implies $\Phi_1 \vee \cdots \vee \Phi_n$. Hence ψ does not imply Φ_i, and χ does not imply $\Phi_1 \vee \cdots \vee \Phi_{i-1} \vee \Phi_{i+1} \vee \cdots \vee \Phi_n$. Hence there is an assignment of truth values to the letters of Φ_i which makes ψ true and Φ_i false, and there is an assignment of truth values to the rest of the alphabet which makes χ true and $\Phi_1 \vee \cdots \vee \Phi_{i-1} \vee \Phi_{i+1} \vee \cdots \vee \Phi_n$ false. Pooling the two assignments (which we can do since the sets of letters concerned are mutually exclusive), we have an assignment which makes ϕ true and $\Phi_1 \vee \cdots \vee \Phi_n$ false. But this is impossible, since ϕ by hypothesis implied $\Phi_1 \vee \cdots \vee \Phi_n$.

THEOREM 8. *If* $\Phi_1 \vee \cdots \vee \Phi_n (n > 1)$ *is irredundant and no two of* $\Phi_1, \cdots, \Phi_n$ *have letters in common, then any simplest normal equivalent of* $\Phi_1 \vee \cdots \vee \Phi_n$ *will be of the form* $\Psi_1 \vee \cdots \vee \Psi_n$ *where* $\Psi_1, \cdots, \Psi_n$ *are equivalent respectively to* $\Phi_1, \cdots, \Phi_n$.

Proof. Let Ψ be a simplest normal equivalent of $\Phi_1 \vee \cdots \vee \Phi_n$. Then Φ_i, for each i, implies Ψ. Now suppose that Φ_i has no letters in common with Ψ. Implication can occur without common letters only in the extreme cases where the implying formula is inconsistent or the implied one is valid. But Φ_i, being normal, is consistent. So Ψ would have to be valid. Then its equivalent $\Phi_1 \vee \cdots \vee \Phi_n$ would be valid, and hence, by Theorem 6, would be simply '$p \vee \bar{p}$' or '$q \vee \bar{q}$' or the like. But by hypothesis this is impossible; for by hypothesis $n > 1$, and therefore $\Phi_1 \vee \cdots \vee \Phi_n$ contains two or more distinct letters. We conclude, therefore, that Φ_i for each i has letters in common with Ψ. Conversely, by Theorems 1 and 7, each clause of Ψ contains letters exclusively of Φ_i for some i. Therefore Ψ has the form $\Psi_1 \vee \cdots \vee \Psi_n$ where Ψ_i, for each i, contains letters exclusively of Φ_i. It remains to show that Φ_i implies Ψ_i and vice versa. We saw that $\Phi_1 \vee \cdots \vee \Phi_n$ is not valid; neither, therefore, is its equivalent $\Psi_1 \vee \cdots \vee \Psi_n$ valid. So there is an assignment $\mathfrak{A}$, of truth values to the letters of $\Psi_1 \vee \cdots \vee \Psi_{i-1} \vee \Psi_{i+1} \vee \cdots \vee \Psi_n$, which makes the latter formula false. Consider now any assignment $\mathfrak{B}$, of truth values to the letters of Φ_i, which makes Φ_i true. We can combine $\mathfrak{B}$ with $\mathfrak{A}$, since the sets of letters concerned are mutually exclusive; and the combined assignment makes Φ_i true and $\Psi_1 \vee \cdots \vee \Psi_{i-1} \vee \Psi_{i+1} \vee \cdots \vee \Psi_n$ false. Since the combined assignment makes Φ_i true, it must make $\Psi_1 \vee \cdots \vee \Psi_n$ true (for this latter is equivalent to $\Phi_1 \vee \cdots \vee \Phi_n$). More particularly then it must make Ψ_i true (for we just noted that it made the rest of $\Psi_1 \vee \cdots \vee \Psi_n$ false). But the letters of Ψ_i receive truth values only from $\mathfrak{B}$, not $\mathfrak{A}$; and $\mathfrak{B}$ was any assignment which makes Φ_i true. Therefore Φ_i implies Ψ_i. An exactly parallel argument, interchanging the roles of $\Phi_1, \cdots, \Phi_n$ with those of $\Psi_1, \cdots, \Psi_n$, shows conversely that Ψ_i implies Φ_i, thus completing the proof of Theorem 8.

Summarized, our results are as follows. We found, to begin with, a fairly rapid method of reducing any normal formula (and therefore any consistent formula) to the extent of locating and cancelling any redundancies. But we found also that an irredundant normal equivalent was not necessarily a simplest normal equivalent. Accordingly, taking a fresh start, we worked out a routine which could be depended upon to reveal a simplest normal equivalent, and indeed all the simplest normal equivalents. This routine, though not unmanageable, turned out to be far more laborious than the method of merely locating and cancelling redundancies. Moreover, the two methods are almost independent. The laborious method of finding simplest normal equivalents depends on a preliminary expansion into a developed normal formula, and this expansion is not affected by any previous cancelling of redundancies. The only way in which the cancelling of redundancies contributes to the ultimate technique is in connection with the auxiliary expedient of separation developed in these last few pages. Clearly it would be desirable to find a quicker way of getting simplest normal equivalents, say by gearing the whole routine to irredundant formulas rather than to developed formulas. I have not seen how to manage this.

It may be useful to note one particular class of normal formulas which can be exempted from the foregoing procedures altogether; *viz.*, those normal formulas in which no one letter occurs both affirmatively and negatively. Such a formula is already reduced to simplest normal form as soon as we have merely deleted those of its clauses that subsume others of its clauses. I have proved this fact elsewhere,* for the case where all letters are affirmative; and the present extension then followed by substitution of negations of letters for letters.

* *Dos teoremas sobre funciones de verdad*, Memoria del Congreso Científico Mexicano, (año de 1951), vol. 1 (*ciencias fisicas y matemáticas*). At press.

THE DIFFERENTIAL EQUATION OF A CONIC AND ITS RELATION TO THE ABERRANCY

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1. Introduction.

(i) *General remarks.* In this paper, $\theta \equiv \tan^{-1}(y')$ is the angle which the tangent at a general point of a plane curve $y=y(x)$ makes with the x-axis; accents and dots denote differentiation with respect to x and θ respectively. The notation $\xi \equiv \phi^2 \equiv \rho^{-2/3}$ is used, where ρ is the radius of curvature; any equation relating